Homework 0

(Do not hand in)

The objective of this homework is to help you check if you have the necessary background for the course. The questions are about basic concepts in linear algebra, probability, optimization, and Python programming. For your reference, you can check the following textbooks:

- **Linear Algebra**: G. Strang, Linear Algebra and Its Applications, 4th Ed.
- **Probability**: S. Chan, Introduction to Probability for Data Science, 1st Ed. (This is my book. You can find it on my website.)
- **Optimization**: S. Boyd and L. Vandenberghe, Convex Optimization, 1st Ed.

Please complete this homework exercise. You do not need to submit the homework. We will have a 30-minutes dry run quiz between Jan-22 and Jan-23. The purpose of the quiz is to show you what to expect in our actual quiz. The dry run quiz will be based on this homework. Both this homework and the dry run quiz will not count towards your grade.

Exercise 1: Linear Algebra

One of the most common problems we ask in multivariate Gaussian is whether the model is valid. This leads to the question of whether the covariance matrix is positive definite or not. So what is positive definite? A real symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is said to be positive definite if you can verify at least one of the following conditions:

(i) $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0$ for all nonzero real nonzero vectors $\boldsymbol{x}$.

(ii) All the eigenvalues of $\boldsymbol{A}$ satisfy $\lambda_i(\boldsymbol{A}) > 0$, where $\lambda_i(\boldsymbol{A})$ denotes the i-th eigenvalue of $\boldsymbol{A}$.

(iii) All the upper left submatrices $\boldsymbol{A}_k$ have positive determinants, i.e., $|\boldsymbol{A}_k| > 0$, for all k.

(iv) All the pivots (without row exchange) satisfy $d_k > 0$. (Check Wikipedia on pivot.)

Now, consider the following matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. For what range of numbers a and b are the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ positive definite?

$$A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{pmatrix}.$$

Exercise 2: Random Variable

In machine learning, it is quite common to handle one or more random variables. Thus, it is necessary to understand how to verify the distribution of some basic random variables. One very useful concept is the moment generating function. The moment generating function of a random variable X is defined as

$$M_X(s) = \mathbb{E}[e^{sX}], \tag{1}$$

where the expectation is taken over X. A few common moment generating functions can be found in my ECE 302 website, under the section Table. For example, if X is a Gaussian random variable with mean μ and variance σ^2, then the moment generating function of X is

$$M_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}.$$

Now, let X and Y be two independent Gaussian random variables. Assume X has mean μ_X and variance ${\sigma_X}^2$, Y has mean μ_Y and variance ${\sigma_Y}^2$. Define a new random variable $Z = X + Y$.

(a) Show that Z is a Gaussian random variable. Hint: Show that Z has a moment generating function having the same form of a Gaussian.

(b) Find the variance of Z.

Exercise 3: Probability

This is an exercise trying to link probability and linear algebra. Let $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. We say that $\boldsymbol{\Sigma}$ has an eigen-decomposition if $\boldsymbol{\Sigma}$ can be written as

$$\boldsymbol{\Sigma} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T, \tag{2}$$

for some unitary matrices $\boldsymbol{U}$ such that $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$, and diagonal matrix $\boldsymbol{\Lambda}$.

Consider a n-dimensional random vector $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$. Let $\boldsymbol{\Sigma} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^T$ be the eigen-decomposition of $\boldsymbol{\Sigma}$. Let $\boldsymbol{Y} = \boldsymbol{U}^T\boldsymbol{X}$. Find the covariance matrix of $\boldsymbol{Y}$, defined below, in terms of $\boldsymbol{\Lambda}$.

$$\text{Cov}(\boldsymbol{Y}) \stackrel{\text{def}}{=} \mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^T] - \mathbb{E}[\boldsymbol{Y}]\mathbb{E}[\boldsymbol{Y}]^T.$$

Exercise 4: Optimization

Optimization plays a major role in this course. The most basic concept in constrained optimization is the notion of Lagrange multiplier. Consider minimizing a function

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned} \tag{3}$$

The Lagrangian of the problem is defined as

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}). \tag{4}$$

The solution of the original problem can be found by seeking a stationary point of the Lagrangian: $\nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$, and $\nabla_{\boldsymbol{\lambda}}\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = 0$.

Solve the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \|\boldsymbol{x} - \boldsymbol{y}\|^2, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned} \tag{5}$$

Express your answer in terms of $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{y}$.

Exercise 5: Python

We will be using Python for our course. To make sure that you at least know some basics about Python, we ask you to write a very simple program. If you do not have Python installed on your computer, you can try Google Colab.

In Python, import `numpy` and `matplotlib.pyplot`. Plot the function

$$f(x) = \frac{1}{1 + e^{-5(x-1)}}, \tag{6}$$

for $-5 \leq x \leq 5$. Make the linewidth of the curve 6 points. Use black color for the curve. Call x-label as `x`, y-label as `f(x)`, and title as `'my plot'`. Make sure you know how to download the plot and save it on your computer (mouse right click the figure, and click save image as).